Tensor of Second Rank :

A set of N^2 functions A^{ij} are said to be the components of a contravariant tensor of rank two if they transform according to

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} A^{ij} \quad \text{—— (11)}$$

under coordinate transformations where $\bar{A}^{\alpha\beta}$ are the components of the tensor in barred coordinate system.

Similarly for covariant tensor of second rank, we have

$$\bar{A}_{\alpha\beta} = \frac{\partial x^{i}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A_{ij} \, , \quad \text{—— (12)}$$

under the coordinate transformation

A set of N^2 functions A^{i}_{j} are said to be components of a tensor of contravariant rank one and covariant rank one (or mixed tensor of rank two) if they transform, under coordinate transformation, according to

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A^{i}_{j} \quad \text{—— (13)}$$

A_{ij} (the covariant tensor of rank two) can be represented by a square matrix

$$A_{ij} \equiv \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & \ddots & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \quad \text{—— (14)}$$

General form for a tensor of arbitrary rank:

$$\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_p}} A^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_q} \quad (15)$$

Where $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are a set of N^{p+q} functions and are said to be components of a tensor of contravariant rank p and covariant rank q. Total rank $(p+q)$.

and α_r (for $1 \le r \le p$), β_r (for $1 \le r \le q$) are free indices each having value between 1 and N, and i_r, l_s are dummy indices with summation over each from 1 to N.

Equality & null tinsor: consider two tensors

$A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$.

If both the tensors have same contravariant and covariant rank and every component of one is equal to the corresponding component of the other, then

$$A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q}. \quad \text{—— (1)}$$

The two tensors have the same contravariant and covariant rank, they are called of the same type

null tensor:- If the N^k components of a tensor of total rank k identically vanish, we say it to be a null tensor.

Addition and substraction of tensor:-

Two tensor of the same type can be added and substracted.

The resultant tensor will have the same rank as the original tensors.

$$C^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} + B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} \quad \text{—— (2)}$$

One can prove that $C^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ is a tensor. Let us write the transformation components of $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ in the form

$$\bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} B^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{--- (3)}$$

Similarly

$$\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_q} \quad \text{--- (4)}$$

Adding (3) and (4)

$$\bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} \left(A^{i_1 \cdots i_p}_{l_1 \cdots l_q} + B^{i_1 \cdots i_p}_{l_1 \cdots l_q}\right) \quad \text{--- (5)}$$

Now writing the sum of the components of the two tensors in the barred coordinate system as

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} . \quad \text{--- (6)}$$

Now from Eq. (5) & (6)

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} C^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{--- (7)}$$

From Eq. (7) it is obvious that $C^{i_1 \cdots i_p}_{l_1 \cdots l_q}$ is a tensor of contravariant rank p and covariant rank q.

Similary we can define substraction of the two tensors $A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$. ~~let substra~~

$$A^{i_1 \cdots i_p}_{j_1 \cdots j_q} - B^{i_1 \cdots i_p}_{j_1 \cdots j_q} = D^{i_1 \cdots i_p}_{j_1 \cdots j_q} \quad \text{--- (8)}$$